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M334 PART 0

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

PART 0 GUIDE TO THE COURSE

DIFFERENTIAL GEOMETRY

THE OPEN UNIVERSITY

Mathematics: A Third Level Course

DIFFERENTIAL GEOMETRY PART 0

O.U. COURSE UNITS COLLECTION
FOR REFERENCE ONLY

GUIDE TO THE COURSE

Prepared by the Course Team

THE OPEN UNIVERSITY PRESS

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BAR CODE No. [illegible]89
X
CLASS No. 516.36 DH
17 SEP 1991
WITHDRAWN FROM STOCK
LOAN CAT NL

The Open University Press, Walton Hall, Milton Keynes

First published 1975

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Produced in Great Britain by
The Open University Press

ISBN 335 05700 4

This text forms part of the correspondence element of an Open University Third Level Course. The complete list of parts in the course is given at the end of this text.

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Further information on Open University courses may be obtained from The Admissions Office, The Open University, P.O. Box 48, Milton Keynes, MK7 6AB.

1.1

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A BRIEF HISTORY OF DIFFERENTIAL GEOMETRY

This is a very brief history of those aspects of differential geometry that are dealt with in this course. It is here to give some historical perspective in a course which developes the modern approach to old questions. Differential geometry is a vast subject and most of the great mathematicians of the last three centuries have made some contribution to it.

Some of the basic problems of differential geometry were already being studied before the introduction of the calculus. For instance, in connection with his study of light, Huygens (1629-95) studied the curvature of certain plane curves. However, the calculus enabled Newton (1647-1727) to make a more detailed study of such curves and Euler (1707-83) was able to study curves on more general surfaces. The study of surfaces themselves was opened up by the use of patches introduced in a restricted way by Monge (1746-1818) and used by Gauss (1777-1855) to describe the curvature of a surface in an intrinsic way. This enabled one to answer such questions as whether it was possible to make a plane map of the globe that retained all the local properties of its geometry.

It was Riemann (1826-66) who saw how the theory of surfaces could be generalized. The study of Riemann surfaces showed finally that Euclidean geometry was not the only possibility. There is now a vast literature on such geometries.

Meanwhile, the theory of curves took a great step forward with the discovery, by Frenet (1816-1900) and Serret (1819-85), of formulas describing the curvature and torsion of curves. Their "method of moving frames" was adapted by Darboux (1842-1907) to describe surfaces. This method was brought to full generality by Cartan (1869-1951), who brought back geometric insight into the profusion of equations thrown up by Riemann and his followers.

A readable introduction to some of the geometric ideas dealt with in this course can be found in:

D. Hilbert and S. Cohn-Vossent: *Geometry and the Imagination* (Chelsea, 1952).

This section on differential geometry contains pictures of several of the surfaces we shall be studying.

BIBLIOGRAPHY AND CONVENTIONS

Bibliography

The set book for this course is Barrett O'Neill: *Elementary Differential Geometry* (Academic Press, 1966). It is essential to have this book: the course is based on it and will not make sense without it.

The set books for M201, M231 and MST 282 are referred to occasionally; they are useful but not essential. They are:

D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins: *An Introduction to Linear Analysis* (Addison-Wesley, 1966).

E.D. Nering: *Linear Algebra and Matrix Theory* (John Wiley, 1970).

M. Spivak: *Calculus,* paperback edition (W.A. Benjamin/Addision-Wesley, 1973).

R.C. Smith and P. Smith: *Mechanics*, SI Edition (John Wiley, 1972).

Conventions

Unreferenced pages and sections denote the set book. Otherwise

O'Neill	denotes the set book;
Text	denotes the correspondence text;
KKOP	denotes *An Introduction to Linear Analysis* by D.L. Kreider, R.G. Kuller, D.R. Ostberg and F.W. Perkins;
Nering	denotes *Linear Algebra and Matrix Theory* by E.D. Nering;
Spivak	denotes *Calculus* by M. Spivak;
Smith	denotes *Mechanics* by R.C. Smith and P. Smith.

Reference to Open University Courses in Mathematics take the form:

Unit M100 22, Linear Algebra I

Unit MST 281 10, Taylor Approximation

Unit M201 16, Euclidean Spaces I: Inner Products

Unit M231 2, Functions and Graphs

Unit MST 282 1, Some Basic Tools.

References to Definitions, Theorems, Lemmas etc. in the set book, O'Neill, are given as in the following example:

Lemma V.4.2 stands for Lemma 4.2 of Chapter V.

Sometimes if O'Neill is referring to something in the same chapter he omits the roman numeral for the Chapter.

PREREQUISITES

The courses that are prerequisites for this course are M201 Linear Mathematics, M231 Analysis, and MST 282 Mechanics and Applied Calculus.

However, this course has been written with very few explicit references to any of these courses. In this section we list those units and sections of the prerequisite courses that we have built on, and this is followed by a brief description of their content. This should enable you to work out whether you are familiar with those ideas and results that we shall be assuming. The bracketed sections below are marked as optional to M231 and so you may not have read them yet. If you want to, do so now, noting only the ideas and results. The details of the definitions and proofs will not be needed and our notation will not be similar.

Prerequisite Sections

M201 Linear Mathematics

Unit 1	Vector Spaces
Unit 2	Linear Transformations
Unit 3	Hermite Normal Form, Section 3.3
Unit 5	Determinants and Eigenvalues
Unit 10	Jordan Normal Form, Section 10.2.1
Unit 12	Linear Functionals and Duality, Section 12.1
Unit 14	Bilinear and Quadratic Forms, Sections 14.1 and 14.2.3
Unit 15	Affine Geometry and Convex Cones, Sections 15.1.1 and 15.1.2
Unit 16	Euclidean Spaces I, Inner Proudcts
Unit 23	The Wave Equation, Section 23.1.1
Unit 24	Orthogonal and Symmetric Transformations, Sections 24.1, 24.3.1 and 24.3.2

M231 Analysis

Unit 2	Functions and Graphs (plus Section 2.7)
Unit 5	Differentiation (plus Section 5.9)
(Unit 7	Applications of the Derivative, Section 7.8)
Unit 10	Some Important Functions

MST 282 Mechanics and Applied Calculus

Unit 1	Some Basic Tools, Sections 1.1. to 1.3
Unit 2	Kinematics, Sections 2.1.1 and 2.2
Unit 7	Work and Energy I, Section 7.1.2

Detailed Prerequisites

M201 Linear Mathematics

This is the major prerequisite for our course. Though we do not often quote results from it, we freely use all the basic ideas and conventions that it introduced.

Unit 1 Vector Spaces This unit introduces the vector space of arrows, R^2 and R^3, which are the basis for our course. You should also know about *linear dependence, bases* and *coordinates*, though we shall deal only with concrete examples in R^3. Ideas such as *dimension* and *subspaces* are implicit in our course but you will not need any explicit results. In fact, the ideas of this unit are basic to all linear mathematics.

Unit 2 Linear Transformations This unit again introduces ideas basic to our course. You will need to know about *linear transformations*, the convention for representing them by *matrices, isomorphisms* and *rank,* though we shall be interested in 2- and 3-dimensional examples only. This is another unit with which to be entirely familiar.

Unit 3 Hermite Normal Form We do not use the technique of Hermite normal forms in our course. However, we do talk about the *rank* of 2×2 and 3×3 matrices, which can be read off the Hermite normal form obtained by the elementary operations in Section 3.3.

Unit 5 Determinants and Eigenvalues *Determinants, eigenvalues* and *eigenvectors* play a small but important part in our course. The cross product of two vectors will be defined in terms of 3×3 determinants and you should be familiar with the formula for expanding such a determinant dealt with in KKOP, pages 682-687, where the effect of column operations is also described. In Section 5.1.5 it is shown that row operations behave similarly and it is these that we shall use. You will need to know only what eigenvectors and eigenvalues are and not in general how to calculate them. We shall be using eigenvector bases as in Section 5.2.4.

Unit 10 Jordan Normal Form In Section 10.2.1 it is shown that a linear transformation is represented by a diagonal matrix with respect to an eigenvector basis.

Unit 12 Linear Functionals and Duality It will be useful to know the definition of *linear functionals,* the *dual space* and a *dual basis.* However, do not worry about any technical results. Since we shall be dealing with R^2 and R^3, in which there is a simple inner (scalar, dot) product, we shall be able to describe dual bases etc. in a concrete way.

Unit 14 Bilinear and Quadratic Forms Again you will find it useful to have met *bilinear forms* and *symmetric bilinear forms.* It will give you some more insight into the course, although the only explicit result we shall use is the quadratic Taylor approximation.

Unit 15 Affine Geometry and Convex Cones Though this unit is not in the spirit of our course, Sections 15.1.1 and 15.1.2 give various ways of describing affine lines and planes which may prove useful to you.

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Unit 5	Determinants and Eigenvalues
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Unit 12	Linear Functionals and Duality, Section 12.1
Unit 14	Bilinear and Quadratic Forms, Sections 14.1 and 14.2.3
Unit 15	Affine Geometry and Convex Cones, Sections 15.1.1 and 15.1.2
Unit 16	Euclidean Spaces I, Inner Proudcts
Unit 23	The Wave Equation, Section 23.1.1
Unit 24	Orthogonal and Symmetric Transformations, Sections 24.1, 24.3.1 and 24.3.2

M231 Analysis

Unit 2	Functions and Graphs (plus Section 2.7)
Unit 5	Differentiation (plus Section 5.9)
(Unit 7	Applications of the Derivative, Section 7.8)
Unit 10	Some Important Functions

MST 282 Mechanics and Applied Calculus

Unit 1	Some Basic Tools, Sections 1.1. to 1.3
Unit 2	Kinematics, Sections 2.1.1 and 2.2
Unit 7	Work and Energy I, Section 7.1.2

Detailed Prerequisites

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Unit 16 Euclidean Spaces I: Inner Products Sections 16.1 to 16.3 are very important to our course. They introduce the *inner* (scalar, dot) *product* of vectors, which we use extensively. You will need to be able to use it to define *length, distance, angle* and *orthonormal bases.* We shall frequently use orthonormal bases, orthonormal expansions and the simple form of the inner product in terms of the coordinates with respect to such a basis, all of which are dealt with in Section 16.4.3. Our course goes over all the details again.

Unit 23 The Wave Equation All you need from this unit is the definition of *partial differention* in Section 23.1.1, though even this will be redefined.

Unit 24 Orthogonal and Symmetric Transformations This unit contains many results that are needed in this course. *Orthogonal transformations* are linear space 'morphisms' and the *orthogonal matrices* representing them are characterized in several important ways, including the result that their inverse is equal to their transpose. We shall also use the result that symmetric transformations can be represented by diagonal matrices with respect to suitable eigenvector bases.

M231 Analysis

Unit 2 Functions and Graphs This unit covers all the basic ideas associated with *graphs* and *functions of one variable* with which you need to be familiar.

The appendix, Section 2.7, is very straightforward and you ought to have a look at it. It introduces ideas that will be developed in this course.

Unit 5 Differentiation This unit introduces the basic definitions of *differentiation* and *curves*. These are ideas which will be developed further in this course. However, our approach will be less theoretical. We shall be interested in using techniques rather than worrying about such problems as differentiability.

We shall also be using *partial* and *directional derivatives* extensively. These are introduced in an appendix, Section 5.9. If you have looked at this before you might like to look at it again, otherwise don't worry.

Unit 7 Applications of the Derivative The only relevant part of this unit is the appendix, Section 7.8, on *implicit functions.* This is straightforward and will help you with Chapter IV of this course. Try to look through it before then but do not bother with any proofs or the sections on contunuity and differentiability.

Unit 10 Some Important Functions This unit introduces all the standard functions that you will need in this course. You will need to know about *polynomials, trigonometric functions,* the *exponential* and *logarthmic functions, hyperbolic functions* and *inverse trigonometric and hyperbolic functions.* Useful relations involving these and their derivatives can be found in the summary, Section 10.10.

MST 282 Mechanics and Applied Calculus

Unit 1 Some Basic Tools Sections 1.1 to 1.3 give a useful introduction to *vectors* and *applied calculus*, with which you should be familiar. We shall use the *scalar, vector* and *triple scalar products* of vectors and some simple forms of the *chain rule* for differentiation.

Unit 2 Kinematics In Section 2.1.1 differentiation of a vector is introduced and in Section 2.2 parametric equations of curves are discussed. These are ideas that will be developed further and so you should be familiar with them.

Unit 7 Work and Energy I Section 7.1.2 contains some very important ideas that we shall develop further and so you should be familiar with them. It deals with *scalar fields, level curves and surfaces, directional derivatives* and the *gradient.* It also contains a simple form of the chain rule for differentiation, for which it gives a proof in Appendix 1.

Handbooks

Each of the prerequisite courses has a handbook which contains some of the definitions to which we refer. However, the most useful parts of the handbooks are those giving lists of the derivatives of standard functions. These are

M231 Handbook: Section 5, Familiar Functions, pages 45-48

MST 282 Handbook: Section 4, Table of Derivatives, page 27.

THE STRUCTURE OF THE COURSE

This course is based on B. O'Neill's book *Elementary Differential Geometry* published by Academic Press. Again we must stress that it is essential to have this book since the course is based almost entirely on it. The course consists of the first six chapters of the book, apart from a few odd sections, and, with one minor exception, you should read the book in the order it is written. We expect you to read it section by section and we have provided a commentary on each relevant section. The commentaries have been bound together chapter by chapter.

There is another way of viewing the structure of the course which might prove useful. All those sections dealing with a particular theme can be thought of as forming a block. We shall show which sets of sections form blocks and also how these blocks are interrelated, that is, which blocks it is essential to have tackled, at least in part, before going onto another block. If you ever become stuck with the sections in one block this will enable you to see if there is any alternative block that you can tackle first.

Sometimes a particular section does not depend on all those sections that have preceded it in that block or on all the sections of preceding blocks. We have included this information in the form of a detailed structure diagram for each block giving all prerequisite blocks and sections.

Each introduction to a section again lists the most important prerequisite sections but we hope that if you refer to the structure diagrams below regularly the overall structure of the course will become clearer.

1. Breakdown into Blocks

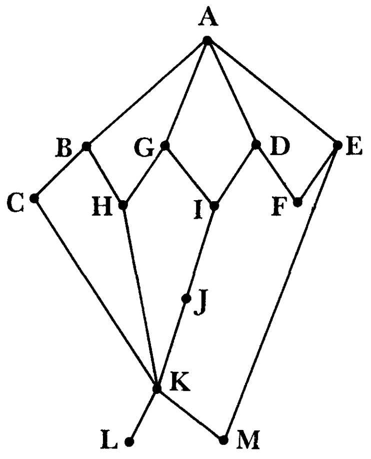
Here is the content of the blocks and the number of units to which they are equivalent, assuming that this is a standard ½ credit course with fourteen teaching units and two gaps.

A.	Calculus and Linear Algebra in E^3 I.1 to I.4, I.7, II.1, II.2, II.5, II.6	3 units
B.	Basic Exterior Calculus in E^3 I.5 and I.6	½ unit
C.	Connection Forms for Frame Fields in E^3 II.7 and II.8	½ unit
D.	Frenet Formulas for Curves in E^3 II.3 and II.4	1 unit
E.	Geometry of E^3 III.1 to III.3	½ unit
F.	Geometry of Curves in E^3 III.4 and III.5	½ unit
G.	Calculus on Surfaces in E^3 IV.1 to IV.3, IV.5(a)	1½ units
H.	Exterior Calculus on Surfaces in E^3 IV.4, IV.5(b), IV.7(a)	1 unit

I.	Shape Operator of a Surface in E^3 V.1 to V.4	2 units
J.	Applications of the Shape Operator V.5 and V.6	1 unit
K.	Structure of Surfaces in E^3 VI.1, VI.2, VI.6	1 unit
L.	Global results on Surfaces in E^3 VI.3, IV.7(b)	½ unit
M.	Geometry of Surfaces in E^3 VI.4, VI.5, VI.8	1 unit

2. Structure of Collection of Blocks

The blocks are related as below, a block is preceded by all those blocks above it to which it is connected.



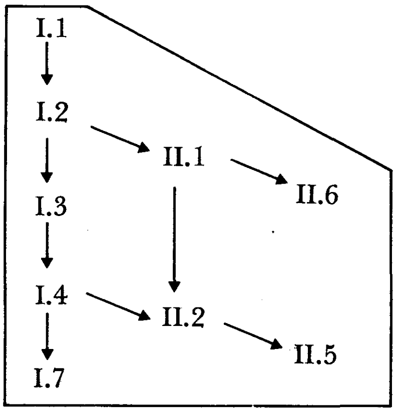
So, for instance, even if you find block **C** too difficult, and on first reading it is quite difficult, there is no reason why you should not press on with **D, E, F, G, H, I, J** as long as you have looked at **C** again before you come to **K** that depends on it.

Each block and all those that precede it tell an interesting story. For instance the line terminating at **J** consists of the blocks **A, D, G, I, J** and describes the shape of surfaces of revolution and interesting curves on surfaces.

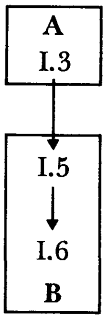
3. Detailed Structure of Blocks and Interconnections

In the following diagrams sections in the same block are grouped together. Arrows indicate how the sections are interconnected. We have shown only how a section depends on those directly preceding it. They in turn may depend on others.

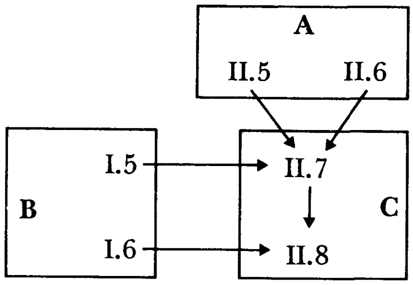
A


I.1
I.2
I.3
I.4
I.7
II.1
II.6
II.2
II.5

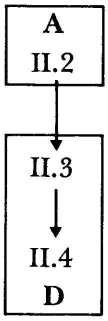
B


A
I.3
I.5
I.6
B

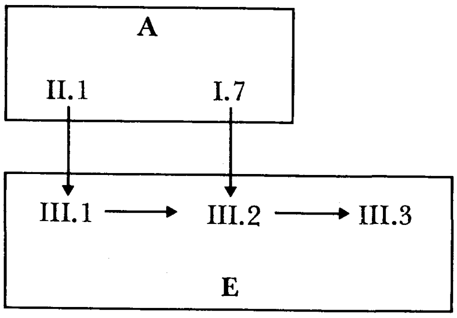
C


A
II.5
II.6
B
I.5
I.6
II.7
II.8
C

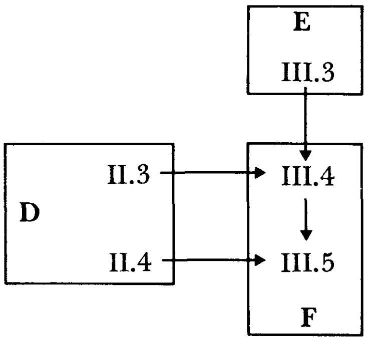
D


A
II.2
II.3
II.4
D

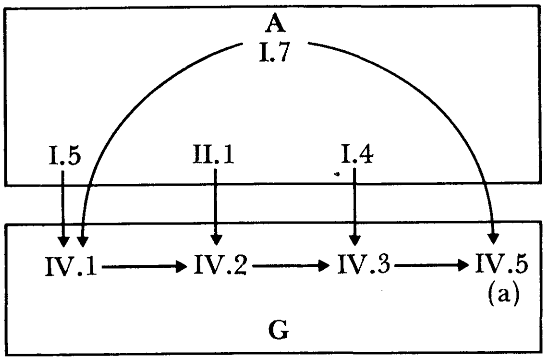
E


A
II.1
I.7
III.1
III.2
III.3
E

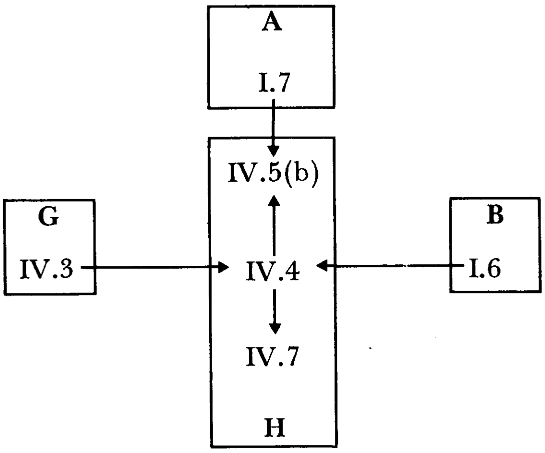
F


E
III.3
D
II.3
II.4
III.4
III.5
F

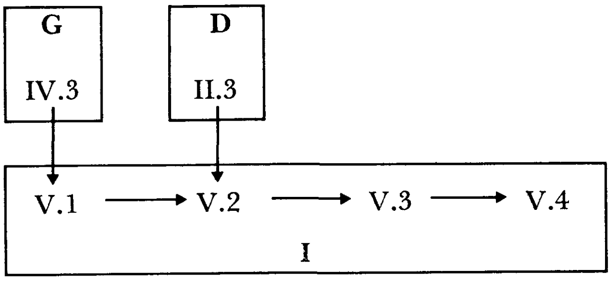
G


A
I.7
I.5
II.1
I.4
IV.1
IV.2
IV.3
IV.5
(a)
G

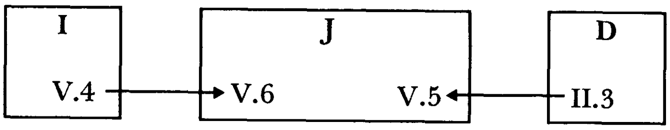
H


A
I.7
IV.5(b)
G
IV.3
IV.4
B
I.6
IV.7
H

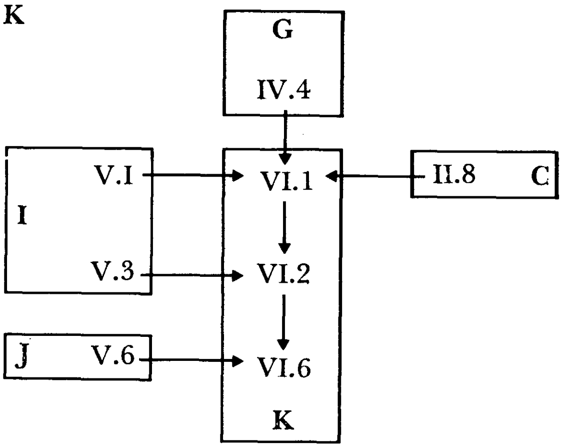
I


G
IV.3
D
II.3
V.1
V.2
V.3
V.4
I

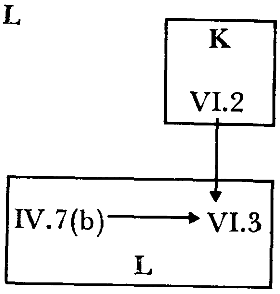
J


I
V.4
J
V.6
V.5
D
II.3

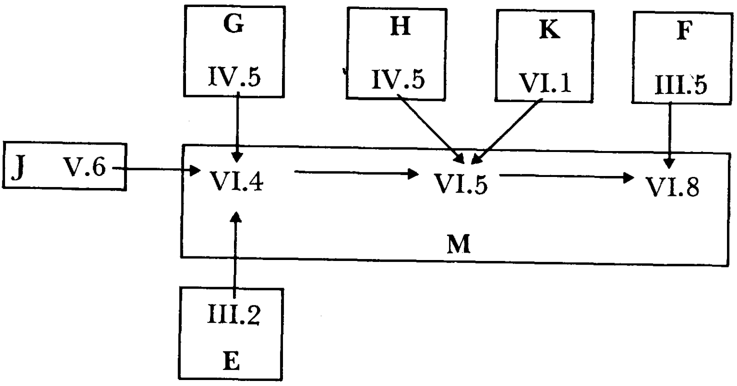
K


G
IV.4
V.I
I
V.3
VI.1
VI.2
VI.6
K
II.8
C
J
V.6

L


K
VI.2
IV.7(b)
VI.3
L

M


G
IV.5
H
IV.5
K
VI.1
F
III.5
J
V.6
VI.4
VI.5
VI.8
M
III.2
E

A POSSIBLE 16-WEEK BREAKDOWN OF THE COURSE

In case you wish to break down the course into 16 weeks of work, the following is a suggested way of doing so.

Week	Sections of O'Neill	Blocks	TMA
1	I.1, I.2, I.3, I.4	A	01
2	I.5, I.6, I.7	A, B	
3	II.1, II.2	A	
4	II.3, II.4	D	02
5	II.5, II.6, II.7, II.8	A, C	
6			
7	III.1, III.2, III.3, III.4, III.5	E, F	
8	IV.1, IV.2	G	03
9	IV.3, IV.4	G, H	
10	IV.5, IV.7(a)	G, H	
11	V.1, V.2, V.3	I	
12	V.4	I	04
13	V.5, V.6	J	
14	VI.1, VI.2, VI.3, IV.7(b)	K, L	
15	VI.4, VI.5, VI.6	K, M	
16	VI.8	M	

WORKING INSTRUCTIONS

This section deals with instructions on how to tackle written material of this course. Of course, we realize that by now that you will probably have worked out which way it is best for you to tackle an Open University text and you may not wish to change successful habits. However, while we were writing our commentaries we had to keep in mind a certain approach to reading them and this was responsible for them being structured in the way they are.

Components

This course has two components. The first is the Set Book, B. O'Neill: *Elementary Differential Geometry*, and the second is the Course Notes. The set book is divided into Chapters which correspond to Parts of the course notes and both of these are divided into Sections indexed in the same way.

Reading a Section

Each section is divided up in the following way.

Introduction

Each section begins with a short introduction. The content of this introduction depends on just how much of an introduction there is in the set book. The introduction tells you the prerequisite sections and possibly some essential results that will be used. It goes on to describe very briefly the content of the section, in terms of those that have gone before, and how this section fits into the overall structure of the course. It will also tell you if it is possible to put this section off till some later time.

Reading Passage

This will usually be an entire section of the set book and it should be read after the introduction. We expect you to read it more than once.

First Reading At the first reading you should try to browse through all the reading passage and obtain an overall view. Look carefully at each definition and try to think of examples. This is a very geometric course so you should try to draw lots of pictures for yourself. Look carefully at any Theorems or Lemmas, making sure you understand all the terms on both sides of each equation, but at this reading do not worry too much about detailed proofs. You should also try and follow the steps in any straightforward worked examples. There are several manipulative techniques that are demonstrated in this way and you will need to master them.

Comments

After completing the reading passage you should next look at the comments we have provided. Though they are related to the text you are not expected to jump backwards and forwards from the reading page to them. In fact some of the comments are quite general and relate to an idea that is developed throughout the section rather than some specific result. You should read these comments in the same spirit as the reading passage.

Additional Text

The Additional Text should be read directly after the Reading Passage and the Comments and approached in exactly the same way. The Additional Text consists of definitions and results that we feel should be discussed at this point and it should be treated as an integral part of the section.

Optional Text

Some sections will contain an Optional Text. These consist either of passages of the text book or of the course notes and develop certain straightforward topics that we consider interesting but peripheral to the main structure of the course. We have included these Optional Texts for those of you who are finding the course enticing and have time to spare; others may omit them without losing anything.

Text Exercises

At the end of the Comments, Additional or Optional Texts you may find some Text Exercises. These are included at this point because they form an integral part of the text. They may deal with some result that has already been referred to or with some result that will be of crucial importance later on.

Second Reading After having worked once through the Reading Passage, Comments and Additional Text you should work through them again but this time in more detail. Once you have grasped the overall structure of the section you need to fill in the finer details. Look up references needed to complete a proof and work through all proofs and examples filling in the missing steps.

Supplementary Comments

It is not essential that you read these comments. They are intended only to help you at the second reading. They are intended to cover those tricky or routine points which are not essential to the development of the subject but may nevertheless cause some concern. Anything not covered by either the Comments or the Supplementary Comments has been omitted because we do not consider it of any importance.

Comments Versus Supplementary Comments The main difference between these two types of comments is that Comments are an integral part of the text and should be included at each reading and during revision, while Supplementary Comments need be looked at only once to clear up tricky points and are not essential to revision.

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Comments Versus Supplementary Comments The main difference between these two types of comments is that Comments are an integral part of the text and should be included at each reading and during revision, while Supplementary Comments need be looked at only once to clear up tricky points and are not essential to revision.

Summary

After mastering as much of the Reading Passage, Comments and Additional Text as you can, you should look through the Summary provided in the Course Notes. Of course if you feel like preparing your own summary and comparing it with ours you might find it a worthwile exercise. Our summary has several subsections. First comes Notation, then Definitions, Results, Examples and Techniques.

Notation We have included only representative examples of each notation introduced in this section.

Definitions and Results We have included in our list of Definitions and Results some that O'Neill has mentioned only in passing but which we feel are important enough to be considered on a par with all the others.

Examples We have included only those examples that are dealt with regularly throughout the course. Important results about these examples are recorded in later Results subsections.

Techniques These techniques could be referred to as the aims of the Sections. For some of the techniques you need to be able to recognize some new object. For others you need to be able to perform some straightforward manipulation, while for others you need to be able to apply some abstract result or reproduce an argument similar to one used in the text.

Exercises

After inspecting the Summary you should try the Exercises without looking at the solutions. The Exercises fall into the following categories.

Text Exercises These were mentioned above.

Technique Exercises These are the most important exercises and are the type we shall expect you to have mastered. We have related each exercise to the relevant techniques so that you know exactly how you ought to tackle it.

Theory Exercises There will be only a few of this type of exercise. They derive some abstract result or slightly extend those dealt with in the Section.

Optional Exercises These are based on or sometimes constitute the Optional Text and so are only for those working through the Optional Texts.

Solutions

The solutions follow the exercises. We have tried to give fairly lengthy solutions to the early exercises, pointing out the reasons for all the steps, but as time goes on we omit reasoning that has been used previously.

Further Exercises

At the end of our commentary on each Chapter there will be a selection of Further Exercises that can be used for self-assessment or revision purposes. They will be either Technique Exercises or Other Recommended Exercises. The techniques needed will be those of the relevant section and each of the other Recommended Exercises will be accompanied by a brief description of how the exercise fits into the course. The Further Exercises will be followed by very brief Solutions.

FURTHER READING

If you are interested in learning more about the subject after you have completed the course, then there are the remaining sections for you to read.

Sections IV.6 and VI.7 deal with integration on a surface in E^3; Section IV.8 introduces manifolds, and Chapter VII applies all the techniques and results of the preceding Chapters to the study of Riemannian Geometry.

O'Neill also provides a short bibliography on page 391. For more differential geometry, at about the same level, you should read the book by T. J. Willmore; for a more advanced approach you should read the book by N. J Hicks, and for applications the book by H. Flanders.

There are a large number of other books now available on differential geometry and it is also the subject of many current research papers. We hope that this course serves you as an introduction to this literature.

ERRATA

The following list comprises the significant mathematical errors we have found in the text of O'Neill. It does not include typographical errors or mistakes where the meaning is nonetheless clear. Errors in exercises and their solutions are generally given not here but where the exercises are mentioned in the correspondence text.

Chapter I

Page 25, last line of text: The coefficient of v_3 should be $(p_2^2 + 2)$.

Page 31, Exercise 6: "φ" in the last line should be "ϑ".

Page 37, line 16: "Corollary 7.6" should be "Theorem 7.5".

Chapter II

Page 54, line 9: The second term of α'' should be $\dfrac{\mathrm{d}^2\alpha_2}{\mathrm{d}t^2}$

Page 80, Exercise 1: "$\nabla_V W$" should be "$\nabla_v W$".

Chapter III

Page 102, line −2: "$\mathrm{q} = F(\mathrm{q})$" should be "$\mathrm{q} = F(\mathrm{p})$".

Page 108, line −8: This should read

$$\text{“}\left(\sum_k c_{jk}\, a_{ik}\right) = \left(\sum_k a_{ik}\, {}^t c_{kj}\right) = A\,{}^t C.\text{”.}$$

Page 108, lines −5, −4: The right-hand side of these should read

$$\begin{aligned} \text{“} &= \det(A\,{}^t C) \\ &= \det A . \det {}^t C = \det A . \det C\text{”.} \end{aligned}$$

Page 120, line −7: The last two "F"s should be removed.

Chapter IV

Page 136, line −5: The final term should be $\sin^2 v$.

Page 148, Lemma 3.8: This should read

> "If M: $g = c$ is a surface in E^3 and if the gradient vector field $\nabla g = \Sigma\, (\partial g/\partial x_i)\, U_i$ (considered only at points of M) is never zero, then it is a non-vanishing normal vector field on the entire surface M."
>
> The first three lines of the proof should be omitted.

Page 157, Exercise 5. The right-hand side of the first equation should be $g'(f(\mathrm{p}))\mathrm{v}_{\mathrm{p}}[f]$.

Chapter V

Page 204, lines 9 and -2: "$T_p(M)$" should be "$\overline{T}_p(M)$" in each case.

Page 210, line -9: The final "$\mathbf{x}_u$" should be "$\mathbf{x}_v$".

Page 214, line -6: "-1" should be "$-1/b^2$".

Page 218, line -4 and diagram: "$T_p(M)$" should be "$\overline{T}_p(M)$" in each case.

Page 241, line -5: The value of each curvature should be multiplied by -1.

Chapter VI

Page 252, line 18: The matrix should be transposed.

Page 297, line 6 of text: The second surface should be

$$M\colon z = \frac{x^2-y^2}{2}.$$

Page 299, line -2: The right-hand side of the equation should be

"$\overline{E}_3(F(\mathrm{p}))$".

DIFFERENTIAL GEOMETRY

- I Calculus on Euclidean Space
- II Frame Fields
- III Euclidean Geometry
- IV Calculus on a Surface
- V Shape Operators
- VI Geometry of Surfaces in E^3